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Transient Markov Chains
with Stationary Measures

By

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Summary

Conditions are given for the existence of nonnegative solutions Q_i of the equations

$$Q_i = \sum_j Q_j P_{ji}, \qquad i = 0, 1, \cdots$$

where the quantities P_{ji} are the transition probabilities, from state j to state i, of a transient Markov chain.

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1. Introduction and Summary. We consider Markov chains x_n, $n = 0, 1, \cdots$, with denumerable states denoted by integers $0, 1, \cdots$. Let $P_{ij} = P_{ij}^1 = \Pr\{x_{n+1} = j \mid x_n = i\}$ and let P_{ij}^n denote the n-step transition probability from i to j, $n = 1, 2, \cdots$. It is assumed throughout that for each i, j there is an n such that $P_{ij}^n > 0$. For other terminology see Feller [5], Chapter 15. (We shall refer to chains satisfying the above assumption as irreducible.)

We shall be concerned with the existence of solutions of the "equations of stationarity,"

$$Q_i = \sum_{j=0}^{\infty} Q_j P_{ji}, \qquad i = 0, 1, \cdots, \tag{1.1}$$

in the case where the chain is transient. A solution will always mean a set of positive numbers satisfying (1.1). (It is readily seen that if a set of nonnegative numbers, not all 0, satisfy (1.1) they must all be strictly positive.) If the chain has finite mean recurrence times it is known that there is a solution whose elements Q_i are a set of probabilities, $\Sigma Q_i = 1$. See [5]. If it is assumed only that the chain is recurrent, Derman showed, [3], that there is a unique (up to a constant multiplier) solution, with $\Sigma Q_i = \infty$ in case the mean recurrence times are infinite. Derman also showed by examples in [4] that if the chain is transient there may or may not be solutions.

In this note we obtain a necessary condition and a sufficient condition for the existence of a solution of (1.1). The author thinks that the sufficient condition is in some sense close to being necessary. The main results are in Theorems 1 and 2. The corollary to Theorem 2 covers a number of cases of interest.

2. A Necessary Condition. We shall use the following terminology:

Definition 1. A path from infinity is a sequence of states i_1, i_2, $\cdots$, not necessarily all distinct but containing infinitely many distinct states, such that

$$\Pr\{x_{n+1} = i_k \mid x_n = i_{k+1}\} > 0, \qquad k = 1, 2, \cdots$$

Definition 2. A simple path from infinity is a path from infinity, all of whose states are distinct.

Theorem 1. In order that (1.1) should have a solution for an irreducible transient chain, it is necessary that there exist a simple path from infinity.

Proof. Suppose (1.1) has a solution $\{Q_i\}$. Define a set of "inverse probabilities" p_{ij} by

$$(2.1) \qquad p_{ij} = P_{ji}Q_j/Q_i, \quad p_{ij}^{\ n} = P_{ji}^{\ n}\, Q_j/Q_i$$

Since $\Sigma_n\, p_{ij}^{\ n} = (Q_j/Q_i)\, \Sigma_n\, P_{ji}^{\ n} < \infty$, the chain defined by the p_{ij} is transient. Let y_n, $n \geq 0$, be the variables of such a chain. It is evident that for almost every sample sequence $y_0, y_1, \cdots$, the quantities $p_{y_0y_1}$, $p_{y_1y_2} \cdots$ are all positive. Moreover,

because of the transient character of the chain, it is true for almost every sequence that no state is visited infinitely often. Therefore, almost every sequence contains a subsequence $y_0', y_1', \cdots$, such that the y_i' are all distinct and $p_{y_i' y_{i+1}'} > 0$. Referring to (2.1) we see that there is thus a simple path from infinity. This completes the proof of Theorem 1.

As an example, consider the renewal process, for which Derman showed directly that there is no solution in the transient case. Here $p_{ij} = 0$ unless $j = 0$ or $i + 1$, and it is evident that there is no simple path from infinity. In the renewal process the state 0 has a special role; every path from infinity must contain it infinitely often. However, it can be shown by examples that even if there is no simple path from infinity, there need not exist any such distinguished state. In fact, an example, which we do not give here, shows that even if no finite set of states has the property that every path from infinity intersects it infinitely often, there need exist no simple path from infinity.

3. Conditions on the Zeros. The condition of Theorem 1 is not sufficient for the existence of a solution to (1.1). Note that this is a condition on the location of the zeros in the matrix (P_{ij}). The following two examples show that no condition on the zeros can be both necessary and sufficient for the existence of a solution in the transient case, since the zeros of the two examples are in the same place and one has a solution while the other does not.

Example 1. Take $P_{i,i+1} = 1 - e^{-i} - e^{-i^3}$, $P_{i,i-1} = e^{-i}$, $P_{io} = e^{-i^3}$, $i = 1, 2, \cdots$; $P_{o1} = 1$. This chain is clearly transient and irreducible. Theorem 2 below can be applied to show that there is a solution to (1.1). We shall not give the details.

Example 2. Take $P_{i,i+1} = 1 - \left(\frac{1}{2}\right)^i$, $P_{i,i-1} = \left(\frac{1}{4}\right)^i$, $P_{io} = \left(\frac{1}{2}\right)^i - \left(\frac{1}{4}\right)^i$, $i = 1, 2, \cdots$; $P_{o1} = 1$. This chain is likewise transient and irreducible. We now prove that in this case (1.1) has no solution.

Proof that there is no solution for Example 2. Suppose that a solution $\{Q_i\}$ exists. First, the Q_i cannot be bounded. For, setting $i = 0$ in (1.1), we have

$$(3.1) \qquad Q_o = \sum_{j=o}^{\infty} Q_j P_{jo} = \sum_{j=o}^{\infty} Q_j P_{jo}^{\ n},$$

$$NQ_o = \sum_{j=o}^{\infty} \sum_{n=1}^{N} Q_j P_{jo}^{\ n}, \qquad N = 1, 2, \cdots$$

If the Q_j were bound by K we would have

$$(3.2) \qquad NQ_o \leq K \sum_{j=o}^{\infty} \sum_{n=1}^{N} P_{jo}^{\ n} < K \sum_{j=o}^{\infty} \sum_{n=1}^{\infty} P_{jo}^{\ n}$$

Now $\Sigma_{n=1}^{\infty} P_{jo}^{\ n}$ is the expected total number of visits to state 0 of a particle starting in state j and is equal to L_{jo} multiplied by the expected number of visits to 0 starting in 0, where L_{jo} is the probability of reaching 0 from j at least once. It

is readily seen that $\sum_j L_{jo} < \infty$, which means that (3.2) cannot hold for all N. Hence the Q_i, if they exist, must be unbounded. Next, observe that equations (1.1) have the form in this example

$$(3.3) \qquad Q_0 = \sum_{j=0}^{\infty} \left[\left(\frac{1}{2}\right)^j - \left(\frac{1}{4}\right)^j\right] Q_j, \quad Q_1 = Q_0 + \left(\frac{1}{4}\right)^2 Q_2,$$

$$(3.4) \qquad Q_i = \left[1 - \left(\frac{1}{2}\right)^{i-1}\right] Q_{i-1} + \left(\frac{1}{4}\right)^{i+1} Q_{i+1}, \quad i = 2, 3, \cdots .$$

Since $1 - \left(\frac{1}{2}\right)^{i-1} + \left(\frac{1}{4}\right)^{i+1} < 1$ for $i \geq 2$, we see from (3.4) that Q_i cannot be as large as $\mathrm{Max}\,(Q_{i-1}, Q_{i+1})$, $i = 2, 3, \cdots$. Since the Q_i are unbounded they must therefore ultimately increase monotonically to infinity. Hence, there is an i such that if $j \geq i$, $Q_j \geq Q_{j-1}$. Then, from (3.4)

$$(3.5) \qquad Q_{j+1} = \frac{Q_j - \left[1 - \left(\frac{1}{2}\right)^{j-1}\right] Q_{j-1}}{(1/4)^{j+1}} \geq \frac{\left(\frac{1}{2}\right)^{i-1} Q_{j-1}}{(1/4)^{j+1}}$$

$$= 2^{j+3} Q_{j-1}, \qquad j = i, \quad i+1, \cdots$$

By repeated application of (3.5) we have, after a little simplification, $Q_{i+2n+1} > 2^{n^2} Q_{i-1}$, $n = 1, 2, \cdots$ This implies that the sum in (3.3) is infinite, a contradiction. Hence (1.1) can have no solution.

4. A Sufficient Condition.

Definitions. Let ${}_kP_{ij}^{\,1} = P_{ij}$, and in general

$$(4.1) \qquad {}_kP_{ij}^{\,n} = \Pr\{x_n = j;\ x_m \neq k,\ 1 \leq m < n \mid x_0 = i\},$$

$$n = 1, 2, \cdots$$

$$(4.2)\qquad L_{ki}(j) = \sum_{r=j}^{\infty}\sum_{n=1}^{\infty} {}_iP_{kr}{}^{n}P_{ri} + P_{ki}$$

$$= \sum_{n=1}^{\infty} \Pr\{x_n = i;\ x_m \neq i,\ 1 \le m < n;\ x_{n-1} \ge j \mid x_0 = k\}$$

$L_{ki}(j)$ is, if $k \ge j$, the probability that a particle, starting at k, will reach i, the first visit being immediately preceded by a state with index $\ge j$. As before, $L_{ki} = L_{ki}(0)$ will denote the probability of reaching i at all, having started from k.

Theorem 2. In order that (1.1) should have a solution for an irreducible transient chain, the following condition is sufficient: there exists an infinite set K of states such that

$$(4.3)\qquad \lim_{\substack{j \to \infty,\\ k \to \infty,\ k \in K}} L_{ki}(j)/L_{ki} = 0, \qquad i = 0, 1, 2, \cdots$$

Corollary. If the chain is transient and irreducible, and if for each i, $p_{ki} = 0$ except for a finite set of values of k, then (1.1) has a solution.

Proof of Theorem 2. In the recurrent case Chung, [2], showed that we can pick an arbitrary state, say 0, and define $Q'_k = \sum_{n=1}^{\infty} {}_0P_{0k}{}^{n}$, obtaining a solution. Here Q'_k is the expected number of visits to k between visits to 0, and we have (note that because of recurrence $Q'_0 = 1$)

$$(4.4)\qquad \sum_{n=1}^{\infty} {}_0P_{0i}{}^{n} = {}_0P_{0i}{}^{1} + \sum_{n=2}^{\infty}\sum_{j=1}^{\infty} {}_0P_{0j}{}^{n-1}P_{ji} = {}_0P_{0i}{}^{1} + \sum_{j=1}^{\infty} Q'_j P_{ji} = \sum_{j=0}^{\infty} Q'_j P_{ji}.$$

since ${}_oP_{oi}{}^1 = P_{oi}{}^1 = Q_o' P_{oi}{}^1$. In the transient case this does not give a solution. It seems reasonable to try instead

$$Q_i'' = \sum_{j=1}^{\infty} P_{ki}{}^n,$$

the expected total number of visits to i, starting at k. It is perhaps better to think of Q_i'' as the average density of particles at i, if there is a source putting one particle per time unit into the system at k. Although Q_i'' is not itself a solution we may hope that it approaches one as the "source" k moves off to infinity. We must also normalize to keep the Q_i within bounds. Hence we define

$$(4.5) \qquad Q_{ki} = \sum_{n=1}^{\infty} P_{ki}{}^n \Big/ \sum_{n=1}^{\infty} P_{ko}{}^n, \qquad i,k = 0, 1, \cdots$$

We shall show that if the conditions of Theorem 2 hold, then there is a sequence $\{k_m\}$ such that

$$(4.6) \qquad \lim_{m \to \infty} Q_{k_m,i} = Q_i, \qquad i = 0, 1, \ldots,$$

where the Q_i are a solution of (1.1).

Definitions. <u>Let</u> θ_{ij}, $i \neq j$, <u>be the probability that the state, given to be initially</u> i, <u>reaches</u> j, <u>before reaching</u> i; <u>let</u> $1 - \phi_{ij}$, $i \neq j$, <u>be the probability that the state, initially</u> i, <u>returns to</u> i <u>without reaching</u> j.

In general, $\theta_{ij} \neq \phi_{ij}$ for transient chains.

Now suppose the state is initially $i \neq j$, and let V be the total number of visits to j which precede any further visit to i. Then clearly

(4.7) $\qquad E(V) = \theta_{ij}/\phi_{ji}$

Hence we have

(4.8) $$Q_{kj} = Q_{ki}\theta_{ij} / \phi_{ji} + \sum_{n=1}^{\infty} {}_iP_{kj}{}^n, \qquad i \neq j.$$

Then

(4.9) $$Q_{ki} = \sum_{n=1}^{\infty} P_{ki}{}^n = P_{ki} + \sum_{n=2}^{\infty} \sum_{r=0}^{\infty} P_{kr}{}^{n-1}P_{ri}$$

$$= P_{ki} + \sum_{r=0}^{j-1} Q_{kr}P_{ri}$$

$$+ \sum_{r=j}^{\infty} \left[Q_{ki} \frac{\theta_{ir}}{\phi_{ri}} + \sum_{n=1}^{\infty} {}_iP_{kr}{}^n\right] P_{ri}, \quad j > i.$$

Dividing both sides of (4.9) by Q_{ko}, and recalling (4.2), we obtain

(4.10) $$\frac{Q_{ki}}{Q_{ko}} = \sum_{r=o}^{j-1} \left(\frac{Q_{kr}}{Q_{ko}}\right) P_{ri} + \frac{Q_{ki}}{Q_{ko}} \sum_{r=j}^{\infty} \left(\frac{\theta_{ir}}{\phi_{ri}}\right) P_{ri}$$

$$+ \frac{L_{ki}(j)}{Q_{ko}}, \qquad j > i.$$

From (4.8) we see that $Q_{kj} / Q_{ki} > \theta_{ij} / \phi_{ji}$; interchanging i and j gives

$$\theta_{ij} / \phi_{ji} < Q_{kj} / Q_{ki} < \phi_{ij} / \theta_{ji}.$$

Hence ratios such as Q_{kr} / Q_{ko} are bounded away from 0 and ∞ by numbers which may depend on r but are independent of k.

Also, $Q_{ki} = L_{ki}(1 + Q_{ii})$. Thus, the condition of Theorem 2 implies that if $k \in K$, $k \to \infty$, $j \to \infty$, then $L_{ki}(j) / Q_{ko} \to 0$ for each i. The above remarks also imply that there is a sequence $\{k_m\}$, m = 1, 2, ..., with $k_m \in K$, such that

$$\lim_{m \to \infty} \frac{Q_{k_m i}}{Q_{k_m 0}} = Q_i, \text{ say,}$$

exists for each i. Since for each i the ratio Q_{ki} / Q_{ko} is bounded, and since

$$\sum_{r=j}^{\infty} (\theta_{ir} / \phi_{ri}) P_{ri}$$

is arbitrarily small for j sufficiently large, the limits Q_i must satisfy (1.1). This concludes the proof of Theorem 2.

5. Remarks. Blackwell, [1] was concerned with the number of bounded solutions for transient chains of the system

$$(5.1) \qquad q_i = \sum_{j=0}^{\infty} P_{ij} q_j .$$

There are some connections between solutions of (1.1) for a given chain and solutions of (5.1) for the inverse chain. However, the author has not obtained any definitive results in this direction.

We leave unsettled the determination of conditions implying that (1.1) has only one solution (up to a constant multiplier). It is clear that there is some flexibility in the choice of the sequence k_m of section 4 and in some cases different sequences obviously lead to different solutions.

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